Solution exercise 1-1b

Bayesian inference in the Gaussian

using the conjugate normal gamma prior

Prior

$$p(\mu, \tau \middle| m_0, s_0^2, \alpha, \beta) = N\left(\mu \middle| m_0, \frac{s_0^2}{\tau}\right) Ga(\tau | \alpha_0, \beta_0)$$

$$= \left(\frac{2\pi s_0^2}{\tau}\right)^{-\frac{1}{2}} \exp\left[-\frac{\tau}{2s_0^2}(\mu - m_0)^2\right] \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \tau^{\alpha_0 - 1} \exp[-\beta_0 \tau]$$

$$= \frac{\beta_0^{\alpha_0}}{(2\pi s_0^2)^{\frac{1}{2}} \Gamma(\alpha_0)} \tau^{\alpha_0 - \frac{1}{2}} \exp[-\beta_0 \tau] \exp\left[-\frac{\tau}{2s_0^2}(\mu - m_0)^2\right]$$

Posterior

$$\frac{\beta_0^{\alpha_0}}{(2\pi \mathrm{s}_0^2)^{\frac{1}{2}}\Gamma(\alpha_0)}\tau^{\alpha_0-\frac{1}{2}}\exp[-\beta_0\tau]\exp\left[-\frac{\tau}{2s_0^2}(\mu-m_0)^2\right]\prod_{n=1}^{N}\frac{\tau^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\exp\left[-\frac{\tau}{2s_0^2}(y_n-\mu)^2\right]$$

$$\frac{\beta_0^{\alpha_0}}{(2\pi s_0^2)^{\frac{1}{2}}\Gamma(\alpha_0)}\tau^{\alpha_0-\frac{1}{2}}\exp[-\beta_0\tau]\exp\left[-\frac{\tau}{2s_0^2}(\mu-m_0)^2\right]\prod_{n=1}^{N}\frac{\tau^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\ \exp\left[-\frac{\tau}{2}(y_n-\mu)^2\right]$$

$$= C\ \tau^{\frac{N}{2}+\alpha_0-\frac{1}{2}}\exp\left[-\tau\left(\beta_0+\frac{1}{2}\sum_{n=1}^{N}y_n^2\right)-\frac{1}{2}\mu^2\tau\left(\frac{1}{s_0^2}+N\right)+\mu\tau\left(\frac{1}{s_0^2}m_0+\sum_{n=1}^{N}y_n\right)\right]$$

Can read off $s_N = \frac{1}{\left(\frac{1}{s_0^2}+N\right)}$, $\alpha_N = \frac{N}{2}+\alpha_0$

- Next, find m_N given s_N

$$\mu\tau\left(\frac{1}{s_0^2}m_0+\sum_{n=1}^{N}y_n\right) = \mu\tau\left(\frac{1}{s_0^2}+N\right)\frac{\left(\frac{1}{s_0^2}m_0+\sum_{n=1}^{N}y_n\right)}{\left(\frac{1}{s_0^2}+N\right)}$$

$$m_N = \frac{\left(\frac{1}{s_0^2}m_0+\sum_{n=1}^{N}y_n\right)}{\left(\frac{1}{s_0^2}+N\right)} = \frac{m_0}{1+s_0^2 N}+\frac{s_0^2\sum_{n=1}^{N}y_n}{1+s_0^2 N} = \frac{m_0+s_0^2\sum_{n=1}^{N}y_n}{1+s_0^2 N}$$

Plug in:

$$C\,\tau^{\frac{N}{2}+\alpha_0-\frac{1}{2}}\exp\left[-\tau\left(\beta_0+\frac{1}{2}\sum_{n=1}^{N}y_n^2\right)+\frac{\tau}{2s_N}m_N^2\right]\exp\left[-\frac{\tau}{2s_N}(\mu-m_N)^2\right]$$

$$C\,\tau^{\alpha_N-\frac{1}{2}}\exp\left[-\tau\left(\beta_0+\frac{1}{2}\sum_{n=1}^{N}y_n^2\right)+\frac{\tau\left(\frac{1}{s_0^2}+N\right)}{2}\left(\frac{\left(\frac{1}{s_0^2}m_0+\sum_{n=1}^{N}y_n\right)}{\left(\frac{1}{s_0^2}+N\right)}\right)^2\right]\exp\left[-\frac{\tau}{2s_N}(\mu-m_N)^2\right]$$

$$C\,\tau^{\alpha_N-\frac{1}{2}}\exp\left[-\tau\left(\beta_0+\frac{1}{2}\sum_{n=1}^{N}y_n^2-\frac{1}{2}\frac{\left(\frac{1}{s_0^2}m_0+\sum_{n=1}^{N}y_n\right)^2}{\left(\frac{1}{s_0^2}+N\right)}\right)\right]\exp\left[-\frac{\tau}{2s_N}(\mu-m_N)^2\right]$$

$$\beta_N=\beta_0+\frac{1}{2}\left(\sum_{n=1}^{N}y_n^2-\frac{\left(\frac{1}{s_0^2}m_0+\sum_{n=1}^{N}y_n\right)^2}{\left(\frac{1}{s_0^2}+N\right)}\right)$$